On the Stability of Continuous-Discontinuous Galerkin Methods for Advection-Diffusion-Reaction Problems

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Abstract

We consider a finite element method which couples the continuous Galerkin method away from internal and boundary layers with a discontinuous Galerkin method in the vicinity of layers. We prove that this consistent method is stable in the streamline diffusion norm if the convection field flows non-characteristically from the region of the continuous Galerkin to the region of the discontinuous Galerkin method. The stability properties of the coupled method are illustrated by numerical experiments.

1 Introduction

It is well known that the standard continuous Galerkin (cG) finite element method exhibits poor stability properties for singularly perturbed problems. In presence of sharp boundary or interior layers, non physical oscillations pollute the numerical approximation throughout the solution domain, and thus stabilisation techniques need to be employed; see [14] for a survey. The discontinuous Galerkin (dG) method offers a framework for the design of finite element methods with good stability properties; see, e.g., [8] for a survey of their development. This is achieved by relaxing the continuity requirements at the inter-element boundaries, where appropriate upwinded numerical fluxes can be employed. One such dG method is the Interior Penalty (IP) method considered herein.

However dG methods require more degrees of freedom (DOFs) compared to the standard cG method. In this work, we investigate if some of these additional DOFs can be removed without affecting the stability properties of the dG method. The motivation of this work is twofold: firstly we wish to improve our understanding of the well-established dG methods in response to the criticism in the increased number of DOFs; secondly we seek insight into the design of more efficient finite element methods, allowing to combine advantages the dG framework with smaller approximation spaces.

Conceptually we envisage in this work a finite element space that lies between the standard continuous and discontinuous Galerkin spaces. We construct such a space by applying standard continuous elements away from any boundary or internal layers (called the cG region) and discontinuous elements in the region of such layers (called the dG

region). Therefore we call this method the *continuous discontinuous Galerkin* (cdG) finite element method.

In the broader sense, methods of this type have been proposed before by Becker, Burman, Hansbo and Larson [4], Perugia and Schötzau [13], and Dawson and Proft [10]. In [4] a globally reduced discontinuous Galerkin method is studied, whose approximation space consists of the continuous piecewise linear functions enriched with piecewise constant functions. In [13], [10] the local discontinuous Galerkin method [9] has been used on the discontinuous region with transmission conditions on the subset of interelement boundaries where continuous and discontinuous elements meet. Our approach is different to [13], [10] as we impose no such conditions beyond those already imposed by the dG method. Removing the transmission conditions makes the approach more natural, but introduces some difficulties as possible over- and undershoots at the interface of the cG and dG regions must be controlled. Indeed, the analysis is limited to the case where the cdG interface is non-characteristic so that jumps across the interface are controlled by the convection term. This allows us to derive a rigorous stabilty bound for the cdG approximation. On the cG region we split the numerical solution into an approximation to the hyperbolic problem which is weakly dependent on the diffusion coefficient ε and an approximation to the remaining part, which depends more strongly on ε but is small in size. Stability of the approximation on the dG region is shown using the approach of Buffa, Hughes and Sangali [5] with some extensions in the manner of Ayuso and Marini [2]. To our knowledge this paper is the first to present a stability result for the proposed cdG method.

Comparisons of our cdG method with various dG and cG methods have been undertaken by Cangiani, Georgoulis and Jensen [7] and Devloo, Forti and Gomes [11]; see also [6] which studies the cdG approximation as the limit of a dG approximation as the jump penalization on interelement boundaries tends to infinity.

The remainder of this work is organized as follows. In Section 2 we formulate the problem, introduce notation, and explain the assumptions we make. Control on the continuous region is considered in several stages in Section 3 and Section 4. Section 5 presents an inf-sup bound on the discontinuous region. The main result of the paper showing control independent of the perturbation parameter ε and mesh size h follows in Section 6. The theoretical results are illustrated in Section 7 by a numerical experiment.

2   The Model Problem and Notation

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. We introduce the model advection-diffusion-reaction (ADR) problem

$$(2.1) \qquad -\varepsilon\Delta u + b(\boldsymbol{x}) \cdot \nabla u + c(\boldsymbol{x})u = f(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega \subset \mathbb{R}^d,$$
$$(2.2) \qquad u = 0 \qquad \text{on } \partial\Omega$$

with constant diffusion coefficient $0 < \varepsilon \le \varepsilon_{\max}$, $b \in [W^{1,\infty}(\Omega)]^d$, $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. For $0 < \varepsilon \ll 1$ the solution to this problem typically exhibits boundary or interior layers.

Unless otherwise stated, we define C throughout as a positive constant, independent of ε and the finite element approximation space and which may be redefined from line to line. C may depend on $\varepsilon_{\max}$. By $a \lesssim b$ we mean $a \le Cb$.

We consider $\mathcal{T}_h$ to be subdivision of Ω into non-overlapping shape regular simplices or hyper-cubes E, which we shall refer to as the triangulation. Denote by $\mathcal{E}_h$ the union of edges e (or faces for $d \ge 3$) of the mesh and the union of internal edges by $\mathcal{E}_h^o$. Define Γ as
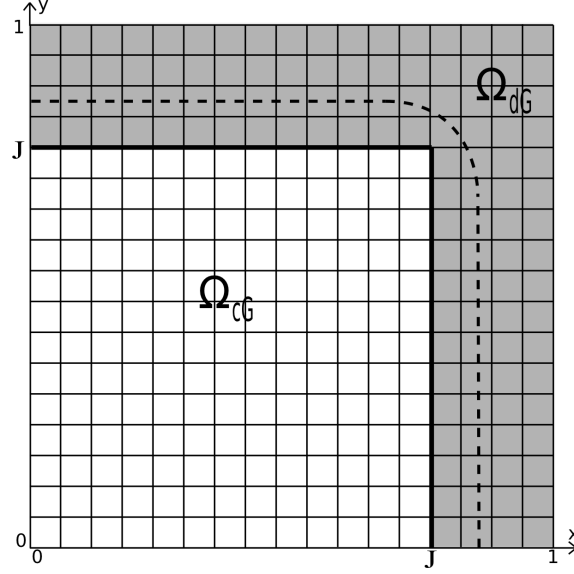
Figure 2.1: An example of a cdG decomposition.

the union of boundary edges, i.e., those lying in $\partial\Omega$. The diameter of an element $E \in \mathcal{T}_h$ is denoted h_E and $h = \max_{E \in \mathcal{T}_h} h_E$. We also denote the mesh function by h_E, thus for $\boldsymbol{x} \in E$ we let $h_E(\boldsymbol{x})$ be equal to the diameter of E. We only consider meshes where $h \leq 1$. Define $h_e := \min(h_{E^-}, h_{E^+})$ for $e \in \mathcal{E}_h^o$ with $e = \bar{E}^+ \cap \bar{E}^-$ for E^+, $E^- \in \mathcal{T}_h$. The mesh is shape regular so there exists $C > 0$ such that for all E we have $h_e \leq \frac{1}{2}(h_{E^+} + h_{E^-}) \leq Ch_e$, and $C \geq 1$ such that $C^{-1}h_{E^-} \leq h_{E^+} \leq Ch_{E^-}$.

We define by Ω-*decomposition* the splitting of Ω into two regions Ω_{cG} and Ω_{dG} such that for the closure $\overline{\Omega} = \overline{\Omega_{\mathrm{cG}} \cup \Omega_{\mathrm{dG}}}$, and we define by $\mathcal{T}_h$-*decomposition* the splitting of $\mathcal{T}_h$ into two sub-meshes $\mathcal{T}_{\mathrm{cG}}$ and $\mathcal{T}_{\mathrm{dG}}$ such that $\mathcal{T}_{\mathrm{cG}} \subset \Omega_{\mathrm{cG}}$ and $\mathcal{T}_{\mathrm{dG}} := \mathcal{T}_h \setminus \mathcal{T}_{\mathrm{cG}}$. By abuse of language, we denoted here by $\mathcal{T}_{\mathrm{cG}}$ not just the sub-mesh but also the region it occupies. Define Γ_{cG} (resp. Γ_{dG}) to be the intersection of Γ with $\overline{\mathcal{T}}_{\mathrm{cG}}$ (resp. $\overline{\mathcal{T}}_{\mathrm{dG}}$). Define $J := \overline{\mathcal{T}}_{\mathrm{cG}} \cap \overline{\mathcal{T}}_{\mathrm{dG}}$ and by convention we say that the edges lying in J are only part of the discontinuous Galerkin skeleton $\mathcal{E}_{\mathrm{dG}}$, the union of faces in $\overline{\mathcal{T}}_{\mathrm{dG}}$, and not part of the continuous Galerkin skeleton defined by $\mathcal{E}_{\mathrm{cG}} := \mathcal{E}_h \setminus \mathcal{E}_{\mathrm{dG}}$.

In Figure 2.1 we illustrate a splitting for a problem where $\Omega = (0,1)^2$ and the solution exhibits layers at $x = 1$ and $y = 1$. The Ω-decomposition is labelled, with the demarcation between the Ω_{cG} and Ω_{dG} regions given by a dashed line. A $\mathcal{T}_h$-decomposition is shown with the $\mathcal{T}_{\mathrm{dG}}$ region shaded and the edges in J marked with a heavy line.

Given a generic scalar field $\nu : \Omega \to \mathbb{R}$, that may be discontinuous across an edge $e = \bar{E}^+ \cap \bar{E}^-$ for E^+, $E^- \in \mathcal{T}_h$, we set $\nu^\pm := \nu|_{E^\pm}$, the interior trace on $E^\pm$ and similarly define $\boldsymbol{\tau}^\pm = \boldsymbol{\tau}|_{E^\pm}$ for a generic vector field $\boldsymbol{\tau} : \Omega \to \mathbb{R}^d$. Define the average and jump for a generic scalar as

$$\{\!\!\{\nu\}\!\!\} := \frac{1}{2}(\nu^+ + \nu^-), \qquad [\![\nu]\!] := \nu^+ n^+ + \nu^- n^-, \qquad \text{on } e \in \mathcal{E}_h^o,$$

and for a generic vector field as

$$\{\!\!\{\boldsymbol{\tau}\}\!\!\} := \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-), \qquad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}^+ \cdot n^+ + \boldsymbol{\tau}^- \cdot n^-, \qquad \text{on } e \in \mathcal{E}_h^o,$$

where $n^\pm$ is the unit outward pointing normal from $E^\pm$ on e. For $e \in \Gamma$ the definitions become

$$\{\!\!\{\nu\}\!\!\} := \nu, \qquad [\![\nu]\!] := \nu n, \qquad \{\!\!\{\boldsymbol{\tau}\}\!\!\} := \boldsymbol{\tau}, \qquad \text{on } e \in \Gamma.$$

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We assume that the sign of $b \cdot n$ is the same for every $\boldsymbol{x} \in e$. Given a vector b denote the inflow and outflow boundaries of Ω by

$$\Gamma^{\text{in}} := \{\boldsymbol{x} \in \partial\Omega : b \cdot n \le 0\}, \qquad \Gamma^{\text{out}} := \{\boldsymbol{x} \in \partial\Omega : b \cdot n > 0\}$$

and for an element

$$\partial^{\text{in}} E := \{\boldsymbol{x} \in \partial E : b \cdot n \le 0\}, \qquad \partial^{\text{out}} E := \{\boldsymbol{x} \in \partial E : b \cdot n > 0\}.$$

On an edge e, we denote by ν^{in} the trace of a function ν, taken from the element which contains e in its inflow boundary.

Definition 2.3. *The cG space is defined by*

$$V_{cG} := \{v \in H^1(\Omega) : \forall E \in \mathcal{T}_h, v|_E \in \mathbb{P}^k\}$$

and the dG space by

(2.4)
$$V_{dG} := \{v \in L^2(\Omega) : \forall E \in \mathcal{T}_h, v|_E \in \mathbb{P}^k\}$$

where $\mathbb{P}^k$ is the space of polynomials of either total (or on quadrilateral meshes partial) degree at most k supported on E.

We seek a finite element space V_{cdG} that lies between the cG and dG spaces in the sense that

(2.5)
$$V_{\text{cG}} \subset V_{\text{cdG}} \subset V_{\text{dG}}.$$

We construct this space by using continuous shape functions away from any boundary or internal layers present in the solution to (2.1) and discontinuous functions in the region of such layers.

Definition 2.6. *The cdG space is defined by*

(2.7)
$$V_{cdG} := \{v \in L^2(\Omega) : \forall E \in \mathcal{T}_h, v|_E \in \mathbb{P}^k; v|_{\Gamma_{cG}} = 0; v|_{\mathcal{T}_{cG}} \in C^0(\overline{\mathcal{T}}_{cG})\}.$$

Throughout we use the same polynomial degree k for V_{cG}, V_{dG} and V_{cdG}.

Let χ be the characteristic function on $\mathcal{T}_{\text{dG}}$, i.e., that defined by

(2.8)
$$\chi := \begin{cases} 1 & \boldsymbol{x} \in \mathcal{T}_{\text{dG}}, \\ 0 & \boldsymbol{x} \in \mathcal{T}_{\text{cG}}. \end{cases}$$

Then define

$$V_{\text{cdG}}(\mathcal{T}_{\text{dG}}) := \{\chi v : v \in V_{\text{cdG}}\}$$

and

$$V_{\text{cdG}}(\mathcal{T}_{\text{cG}}) := \{(1 - \chi)v : v \in V_{\text{cdG}}\}.$$

In order to ensure a unique solution we make the following assumption.

Assumption 2.9. *We assume*

(2.10)
$$r(\boldsymbol{x}) := c(\boldsymbol{x}) - \frac{1}{2}\nabla \cdot b(\boldsymbol{x}) \ge \rho > 0 \qquad \forall \boldsymbol{x} \in \Omega,$$

for some $\rho \in \mathbb{R}$.

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Definition 2.11. *We define the* local mesh Péclet number *to be* $\|b\|_{L^\infty(E)} h_E/(2\varepsilon)$, *see* [14].

We consider meshes in the pre-asymptotic regime by making the following assumption.

Assumption 2.12. *We assume that for $\varepsilon = \varepsilon_{\max}$ and every $E \in \mathcal{T}_h$ the local mesh Péclet number is greater than $\sqrt{h_E}$. Moreover, we require $\max\{\|h_E/b\|_{L^\infty(\mathcal{T}_{dG})}, h_E\} \leq 1$.*

As a consequence we have

$$(2.13) \qquad \varepsilon \leq \varepsilon_{\max} < \frac{1}{2} \min_{E \in \mathcal{T}_h} h_E^{3/2} \|b\|_{L^\infty(\Omega)}.$$

This assumption, for a fixed b, restricts the refinement of the triangulation for a given ε. If we allowed $h \to 0$ for fixed $\varepsilon > 0$ any layers would be resolved by the mesh and in the limit we would not see the non-physical oscillations associated with the cG approximation. We return to this question in Remark 6.6.

To characterise admissible Ω-decompositions of the mesh we introduce the *reduced* problem:

$$(2.14) \qquad \begin{aligned} b \cdot \nabla u_0 + c u_0 &= f \quad \text{on } \Omega, \\ u_0 &= 0 \quad \text{on } \Gamma^{\text{in}}. \end{aligned}$$

Further, we define $u_\varepsilon := u - u_0$, where u is the solution to the ADR problem (2.1).

The Ω-decomposition is chosen such that u_ε and u_0 have additional regularity on Ω_{cG}. In general we do not expect that $u_0 \in H^2(\Omega)$, even if we place higher regularity requirements on f, see, e.g., [3] and the references therein.

Assumption 2.15. *The set $\Omega_{cG} \subset \Omega$ is chosen such that $u_0 \in H^2(\Omega_{cG})$ and $\|u_\varepsilon\|_{H^2(\Omega_{cG})}$ is bounded uniformly in ε, that is for every $0 < \varepsilon \leq \varepsilon_{\max}$*

$$(2.16) \qquad \|u_\varepsilon\|_{H^2(\Omega_{cG})} \lesssim 1.$$

2.1 Decoupled and Standard Formulations

We define by ∇_h the elementwise gradient operator. We discretize the advection term by

$$(2.17) \quad \mathcal{B}_a(w, \hat{w}) := \sum_{E \in \mathcal{T}_h} \int_E (b \cdot \nabla_h w)\hat{w} \, d\boldsymbol{x} - \sum_{e \in \mathcal{E}_h^o} \int_e b \cdot [\![w]\!] \, \hat{w}^{\text{in}} \, ds - \sum_{e \in \Gamma^{\text{in}}} \int_e (b \cdot n) w \, \hat{w} \, ds,$$

and the reaction term by

$$(2.18) \qquad \mathcal{B}_r(w, \hat{w}) := \sum_{E \in \mathcal{T}_h} \int_E c w \hat{w} \, d\boldsymbol{x}.$$

The advection and reaction parts will frequently occur together and so for brevity we also define $\mathcal{B}_{ar}(w, \hat{w}) := \mathcal{B}_a(w, \hat{w}) + \mathcal{B}_r(w, \hat{w})$.

For the diffusion term, besides the standard symmetric interior penalty method we also present a modified scheme, which we call the *decoupled* method. We refer to [1] for a comprehensive study of the interior penalty method and alternative discretizations. For $w, \hat{w} \in V_{\text{dG}}$ the decoupled method is defined by

$$(2.19) \qquad \begin{aligned} \tilde{\mathcal{B}}_d(w, \hat{w}) := &\sum_{E \in \mathcal{T}_h} \int_E \nabla_h w \cdot \nabla_h \hat{w} \, d\boldsymbol{x} \\ &+ \sum_{e \in \mathcal{E}_h \setminus J} \int_e \sigma h_e^{-1} [\![w]\!] \cdot [\![\hat{w}]\!] - (\{\!\{\nabla_h w\}\!\} \cdot [\![\hat{w}]\!] + \{\!\{\nabla_h \hat{w}\}\!\} \cdot [\![w]\!]) \, ds, \end{aligned}$$

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yielding

$$(2.20) \qquad \tilde{\mathcal{B}}_\varepsilon(w, \hat{w}) := \varepsilon \tilde{\mathcal{B}}_d(w, \hat{w}) + \mathcal{B}_a(w, \hat{w}) + \mathcal{B}_r(w, \hat{w}).$$

Here $\sigma > 0$ is a discontinuity penalization parameter. With this formulation there is no control on the fluxes across J (hence the name decoupled). We recall that the standard symmetric Interior Penalty (IP) method is given by

$$\mathcal{B}_d(w, \hat{w}) := \tilde{\mathcal{B}}_d(w, \hat{w}) + \sum_{e \in J} \int_e \sigma h_e^{-1} [\![w]\!] \cdot [\![\hat{w}]\!] - (\{\!\{\nabla_h w\}\!\} \cdot [\![\hat{w}]\!] + \{\!\{\nabla_h \hat{w}\}\!\} \cdot [\![w]\!]) \, \mathrm{d}s,$$

and

$$\mathcal{B}_\varepsilon(\hat{w}, w) := \varepsilon \mathcal{B}_d(\hat{w}, w) + \mathcal{B}_a(\hat{w}, w) + \mathcal{B}_r(\hat{w}, w).$$

When restricted to the cdG space, the decoupled and standard IP forms become the bilinear form for the standard cG method on the continuous region. We first first analyse the stability properties of the decoupled formulation and infer stability for the standard IP method from a perturbation argument.

We introduce the following mesh dependent norm for $w \in V_{\mathrm{dG}}$:

$$(2.21) \qquad \||w\||^2 := \|w\|_d^2 + \|w\|_{ar}^2,$$

where

$$\|w\|_d^2 := \sum_{E \in \mathcal{T}_h} \varepsilon |w|_{H^1(E)}^2 + \sum_{e \in \mathcal{E}_h} \varepsilon \sigma h_e^{-1} \|[\![w]\!]\|_{L^2(e)}^2,$$

and

$$\|w\|_{ar}^2 = \|r^{1/2} w\|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{2} \||b \cdot n|^{1/2} [\![w]\!]\|_{L^2(e)}^2,$$

with r defined in (2.10).

Recall that for the symmetric interior penalty method the parameter σ is selected independently of the $\mathcal{T}_h$ such that $\mathcal{B}_d$ is positive definite with a coercivity constant which is also independent of $\mathcal{T}_h$. We assume that σ is such that for all $w \in V_{\mathrm{dG}}$:

$$(2.22) \qquad \|\{\!\{\nabla_h w\}\!\} \cdot [\![w]\!]\|_{L^1(\mathcal{E}_h)} \le \frac{1}{2} \|\nabla_h w\|_{L^2(\Omega)} \|\sqrt{\sigma/h_e} [\![w]\!]\|_{L^2(\mathcal{E}_h)}.$$

Then, by Young's inequality, we have

$$\frac{1}{2} \|w\|_d^2 \le \mathcal{B}_d(w, w).$$

We adopt for $\tilde{\mathcal{B}}_d$ the same σ as for $\mathcal{B}_d$.

We introduce a projection operator following the presentation of [2] which allows us to consider non-constant b. For polynomial degree $k \ge 0$ consider the L^2-orthogonal projection $\Pi_{\mathcal{D}} : L^2(\Omega) \to V_{\mathrm{cdG}}(\mathcal{T}_{\mathrm{dG}})$ defined by

$$(2.23) \qquad \int_\Omega \Pi_{\mathcal{D}}(v) w \, \mathrm{d}\boldsymbol{x} = \int_\Omega v w \, \mathrm{d}\boldsymbol{x} \qquad \forall w \in V_{\mathrm{cdG}}(\mathcal{T}_{\mathrm{dG}}).$$

In particular $\Pi_{\mathcal{D}}(v)|_{\mathcal{T}_{\mathrm{cG}}} = 0$. Furthermore, for all elements E of the mesh

$$(2.24) \qquad \|\Pi_{\mathcal{D}}(v)\|_{L^2(E)} \le \|v\|_{L^2(E)} \qquad \forall v \in L^2(E).$$

As $\Pi_{\mathcal{D}}(v) \in V_{\mathrm{cdG}}(\mathcal{T}_{\mathrm{dG}})$ we have for all $E \in \mathcal{T}_h$ the inverse inequality

$$(2.25) \qquad |\Pi_{\mathcal{D}}(b \cdot \nabla_h v)|_{H^1(E)} \lesssim h_E^{-1} \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)},$$

and using a trace inequality we have

$$(2.26) \qquad \sum_{e \in \mathcal{E}_h} \|[\![\Pi_{\mathcal{D}}(b \cdot \nabla_h v)]\!]\|_{L^2(e)}^2 \lesssim \sum_{E \in \mathcal{T}_h} h_E^{-1} \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2.$$

Define the *streamline* norm by

$$(2.27) \qquad \|v\|_S^2 := \|\|v\|\|^2 + \sum_{E \in \mathcal{T}_h} \tau_E \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2,$$

where τ_E is defined by

$$(2.28) \qquad \tau_E := \tau \min \left\{ \frac{h_E}{\|b\|_{L^\infty(E)}}, \frac{h_E^2}{\varepsilon} \right\},$$

and τ is a positive number at our disposal.

Definition 2.29. *A decoupled cdG approximation to* (2.1) *is defined as* $\tilde{v}_h \in V_{cdG}$ *satisfying*

$$(2.30) \qquad \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_h, v) = \int_\Omega f v \, \mathrm{d}\boldsymbol{x} \qquad \forall v \in V_{cdG}.$$

Definition 2.31. *A cdG approximation to* (2.1) *is defined as* $v_h \in V_{cdG}$ *satisfying*

$$(2.32) \qquad \mathcal{B}_\varepsilon(v_h, v) = \int_\Omega f v \, \mathrm{d}\boldsymbol{x} \qquad \forall v \in V_{cdG}.$$

We require that b points on J non-characteristically from $\mathcal{T}_{\mathrm{cG}}$ to $\mathcal{T}_{\mathrm{dG}}$.

Assumption 2.33. *The* $\mathcal{T}_h$*-decomposition is such that for every* $e \in J$

$$(2.34) \qquad \frac{1}{4}(b(\boldsymbol{x}) \cdot n^{\mathcal{C}})|_e > \varepsilon_{\max} \frac{\sigma}{h_e^{3/2}} \qquad \forall \boldsymbol{x} \in e,$$

where $n^{\mathcal{C}}$ *represents the unit normal pointing from* $\mathcal{T}_{cG}$ *to* $\mathcal{T}_{dG}$.

Observe that the scaling between ε and h mirrors that of Assumption 2.12.

Theorem 2.35. *On* V_{dG} *the bilinear forms* $\tilde{\mathcal{B}}_\varepsilon$ *and* $\mathcal{B}_\varepsilon$ *are coercive with respect to* $\|\|w\|\|$:

$$(2.36) \qquad \frac{1}{4}\|\|w\|\|^2 \leq \tilde{\mathcal{B}}_\varepsilon(w, w), \qquad \frac{1}{4}\|\|w\|\|^2 \leq \mathcal{B}_\varepsilon(w, w), \qquad w \in V_{dG}.$$

Proof. For the advection and reaction terms using integration by parts we have

$$(2.37) \qquad \mathcal{B}_{ar}(w, w) = \|r^{1/2} w\|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}_h} \int_e \frac{1}{2} |b \cdot n| [\![w]\!] \cdot [\![w]\!] \, \mathrm{d}s.$$

For the diffusion term it follows from Equation 2.22 and Young's inequality that

$$\tilde{\mathcal{B}}_d(w, w) + \int_J \frac{\sigma}{h_e} [\![w]\!] \cdot [\![w]\!] \, \mathrm{d}s \geq \frac{1}{2} \|w\|_d^2, \qquad \mathcal{B}_d(w, w) \geq \frac{1}{2} \|w\|_d^2.$$

Combing the last inequality with (2.37), the result now follows with Assumption 2.33. $\quad\square$

It follows that $\tilde{v}_h$ and v_h exist and are unique. The following final assumption permits the use of an inverse inequality on the continuous Galerkin region.

Assumption 2.38. *The mesh $\mathcal{T}_{cG}$ is quasi-uniform.*

It is convenient to denote the mesh-size on $\mathcal{T}_{cG}$ by $h_{\mathcal{T}_{cG}} = \|h_E\|_{L^\infty(\mathcal{T}_{cG})}$.

The main result of this work is Theorem 6.2, which states that the cdG approximation is stable in the streamline diffusion norm whenever Assumptions 2.9, 2.12, 2.15, 2.33 and 2.38 are satisfied.

In order to prove this result we establish first two separate stability bounds: We define $\tilde{v}_\varepsilon, \tilde{v}_0 \in V_{cdG}$ by the condition that for all $v \in V_{cdG}$

$$(2.39) \qquad\qquad \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_\varepsilon, v) = \tilde{\mathcal{B}}_\varepsilon(u_\varepsilon, v),$$

$$(2.40) \qquad\qquad \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_0, v) = \tilde{\mathcal{B}}_\varepsilon(u_0, v).$$

Observe that by linearity of the decoupled cdG method we have $\tilde{v}_\varepsilon + \tilde{v}_0 = \tilde{v}_h$. In Section 3 we derive a bound for the decoupled cdG approximation $\tilde{v}_\varepsilon$ to u_ε on $\mathcal{T}_{cG}$; in Section 4 we obtain a bound for the decoupled cdG approximation $\tilde{v}_0$ to u_0 on $\mathcal{T}_{cG}$. In Section 5 we establish an inf-sup condition with streamline control on $\mathcal{T}_{dG}$. Finally, in Section 6 we combine these results to show stability of the decoupled and then of the standard cdG approximation.

3 Bounds on the $\tilde{v}_\varepsilon$ Component on $\mathcal{T}_{cG}$

We introduce the projection operator of Scott and Zhang, [15] [12, Section 1.6.2].

Lemma 3.1 (Scott-Zhang Projection). *The Scott-Zhang operator $\mathcal{SZ}_h : W^{l,p}(\Omega) \to V_{cG}$ is a mapping with the following properties: For $l > \frac{1}{2}$ there exists a $C_{sz} > 0$ such that for all $0 \le m \le \min(1, l)$*

$$(3.2) \qquad\qquad \|\mathcal{SZ}_h(v)\|_{H^m(\mathcal{T}_{cG})} \le C_{sz} \|v\|_{H^l(\mathcal{T}_{cG})} \qquad \forall v \in H^l(\mathcal{T}_{cG}),$$

and provided $l \le k + 1$ for all $E \in \mathcal{T}_{cG}$ and $0 \le m \le l$ we have the approximation

$$(3.3) \qquad\qquad \|v - \mathcal{SZ}_h(v)\|_{H^m(E)} \le C_{sz} h_E^{l-m} |v|_{H^l(\Delta_E)} \qquad \forall v \in H^l(\Delta_E).$$

where Δ_E is the node patch of E, i.e., the set of cells in $\mathcal{T}_{cG}$ sharing at least one vertex with E.

Theorem 3.4. *The decoupled cdG approximation $\tilde{v}_\varepsilon$ is stable on the $\mathcal{T}_{cG}$ region in the sense that*

$$(3.5) \qquad\qquad \|\tilde{v}_\varepsilon\|_{H^1(\mathcal{T}_{cG})} \lesssim 1.$$

Proof. We pick the *auxiliary* solution $v_\mathcal{A} \in V_{cdG}$ as follows: On $\mathcal{T}_{cG}$, define $v_\mathcal{A}$ to be the Scott-Zhang projection of u_ε; and on $\mathcal{T}_{dG}$ to be the dG approximation with boundary conditions given by $\mathcal{SZ}_h(u_\varepsilon)$ on J and 0 on Γ_{dG}, i.e.,

$$v_\mathcal{A} = \mathcal{SZ}_h(u_\varepsilon) \quad \text{on } \mathcal{T}_{cG},$$
$$\tilde{\mathcal{B}}_\varepsilon(v_\mathcal{A}, v) = \tilde{\mathcal{B}}_\varepsilon(u_\varepsilon, v) \quad \forall v \in V_{cdG}(\mathcal{T}_{dG}).$$

Set $\eta := u_\varepsilon - v_\mathcal{A}$ and $\xi := v_\mathcal{A} - \tilde{v}_\varepsilon$, so $\eta + \xi = u_\varepsilon - \tilde{v}_\varepsilon$. Notice that $\xi \in V_{\mathrm{cdG}}$. The Galerkin orthogonality expressed by (2.39) and Theorem 2.35 give

$$(3.6) \qquad \tfrac{1}{4}\vert\!\vert\!\vert\xi\vert\!\vert\!\vert \leq \tilde{\mathcal{B}}_\varepsilon(\xi, \xi) = -\tilde{\mathcal{B}}_\varepsilon(\eta, \xi) = -\tilde{\mathcal{B}}_\varepsilon(\eta, \xi - \chi\xi),$$

where χ is defined in (2.8). Note that $\xi - \chi\xi$ is continuous except on J where $[\![\xi - \chi\xi]\!] = \xi^{\mathcal{C}} \cdot n^{\mathcal{C}}$ and $\{\!\{\xi - \chi\xi\}\!\} = \frac{1}{2}\xi^{\mathcal{C}}$, where the superscript $\mathcal{C}$ indicates the trace taken from the continuous Galerkin side of J.

We examine each term of $\tilde{\mathcal{B}}_\varepsilon$ in turn. For the diffusion parts we use Young's inequality

$$-\tilde{\mathcal{B}}_d(\eta, \xi - \chi\xi) \leq 2\vert\eta\vert^2_{H^1(\mathcal{T}_{\mathrm{cG}})} + \tfrac{1}{8}\vert\xi\vert^2_{H^1(\mathcal{T}_{\mathrm{cG}})}.$$

For the advection term we use Assumption 2.33 which ensures that flux terms on J are zero as the upwind value of $\xi - \chi\xi$ vanishes. With Young's inequality we have

$$-\mathcal{B}_a(\eta, \xi - \chi\xi) \leq \tfrac{4}{\rho}\Vert b \cdot \nabla_h \eta\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})} + \tfrac{\rho}{16}\Vert\xi\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})},$$

where ρ is defined in (2.10). Finally for the reaction term

$$-\mathcal{B}_r(\eta, \xi - \chi\xi) \leq \tfrac{4}{\rho}\Vert c\Vert^2_{L^\infty(\Omega)}\Vert\eta\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})} + \tfrac{\rho}{16}\Vert\xi\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})}.$$

Using the previous three results, (3.6), the definition of the norm (2.21), and Lemma 3.1 we gather ξ terms on the left hand side to show, with $h_{\mathcal{T}_{\mathrm{cG}}} = \Vert h_E\Vert_{L^\infty(\mathcal{T}_{\mathrm{cG}})}$,

$$\tfrac{1}{8}\vert\!\vert\!\vert\xi\vert\!\vert\!\vert^2 \leq 2\varepsilon\vert\eta\vert^2_{H^1(\mathcal{T}_{\mathrm{cG}})} + \tfrac{4}{\rho}\Vert b \cdot \nabla_h \eta\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})} + \tfrac{4}{\rho}\Vert c\Vert^2_{L^\infty(\Omega)}\Vert\eta\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})}$$
$$(3.7) \qquad\qquad \lesssim (\varepsilon h^2_{\mathcal{T}_{\mathrm{cG}}} + h^2_{\mathcal{T}_{\mathrm{cG}}} + h^4_{\mathcal{T}_{\mathrm{cG}}})\Vert u_\varepsilon\Vert^2_{H^2(\Omega_{\mathrm{cG}})} \lesssim h^2_{\mathcal{T}_{\mathrm{cG}}}$$

where in the final step we have used (2.16). As $\rho > 0$ we may use (3.7) and an inverse inequality to show

$$(3.8) \qquad \Vert\xi\Vert^2_{H^1(\mathcal{T}_{\mathrm{cG}})} \lesssim h^{-2}_{\mathcal{T}_{\mathrm{cG}}}\Vert\xi\Vert^2_{L^2(\mathcal{T}_{\mathrm{cG}})} \lesssim h^{-2}_{\mathcal{T}_{\mathrm{cG}}}\vert\!\vert\!\vert\xi\vert\!\vert\!\vert^2 \lesssim 1.$$

Assumption 2.15 and (3.2) give $\Vert\tilde{v}_\varepsilon\Vert^2_{H^1(\mathcal{T}_{\mathrm{cG}})} \lesssim 1$. $\qquad\qquad\qquad\square$

4 Bounds on the $\tilde{v}_0$ Component on $\mathcal{T}_{\mathbf{cG}}$

We now pick the auxiliary solution $v_\mathcal{A}$ as follows: On $\mathcal{T}_{\mathrm{dG}}$ let it be u_0 and on $\mathcal{T}_{\mathrm{dG}}$ be the dG approximation to u_0 with boundary conditions given by u_0 on $\Gamma_{\mathrm{dG}} \cup J$, i.e.,

$$(4.1) \qquad\qquad\qquad v_\mathcal{A} = u_0 \qquad \text{on } \mathcal{T}_{\mathrm{cG}},$$
$$(4.2) \qquad\qquad \tilde{\mathcal{B}}_\varepsilon(v_\mathcal{A}, v) = \tilde{\mathcal{B}}_\varepsilon(u_0, v) \qquad \forall v \in V_{\mathrm{cdG}}(\mathcal{T}_{\mathrm{dG}}).$$

Lemma 4.3. *We have for all $v \in V_{cdG}$ that $\tilde{\mathcal{B}}_\varepsilon(v_\mathcal{A}, v) = \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_0, v)$.*

Proof. Fix $v \in V_{\mathrm{cdG}}$. Then using (2.40)

$$\tilde{\mathcal{B}}_\varepsilon(\tilde{v}_0, v) = \tilde{\mathcal{B}}_\varepsilon(u_0, v) = \tilde{\mathcal{B}}_\varepsilon(u_0, v - \chi v) + \tilde{\mathcal{B}}_\varepsilon(u_0, \chi v)$$

where χ is defined in (2.8). Observe that $\tilde{\mathcal{B}}_\varepsilon(u_0, \chi v) = \tilde{\mathcal{B}}_\varepsilon(v_\mathcal{A}, \chi v)$ by (4.2). Notice that $v - \chi v$ and u_0 are continuous on $\mathcal{T}_{\mathrm{cG}}$. Recall that no integral over J appears in the definition of $\tilde{\mathcal{B}}_d$. For $\tilde{\mathcal{B}}_a(\tilde{v}_0, v - \chi v)$, the integral over J vanishes since the value of $(v - \chi v)^{\mathrm{in}}$ is zero because of Assumption 2.33. Therefore $\tilde{\mathcal{B}}_\varepsilon(\tilde{v}_0, v - \chi v) = \tilde{\mathcal{B}}_\varepsilon(u_0, v - \chi v) = \tilde{\mathcal{B}}_\varepsilon(v_\mathcal{A}, v - \chi v)$. $\quad\square$

Lemma 4.4. *We have* $\|\tilde{v}_0\|_{H^1(\mathcal{T}_{cG})} \lesssim 1$.

Proof. Define $\tilde{v}_\pi$ to be

$$\tilde{v}_\pi := \begin{cases} \mathcal{SZ}_h(u_0) & \text{on } \mathcal{T}_{cG}, \\ v_{\mathcal{A}} & \text{on } \mathcal{T}_{dG}, \end{cases}$$

and let $\eta := v_{\mathcal{A}} - \tilde{v}_\pi$, $\xi := \tilde{v}_\pi - \tilde{v}_0$. With these definitions $\eta + \xi = v_{\mathcal{A}} - \tilde{v}_0$, $\eta|_{\mathcal{T}_{dG}} = 0$ and ξ and η are continuous on $\mathcal{T}_{cG}$. Then using Lemma 4.3 we have

$$\begin{aligned} \tfrac{1}{4}\|\|\xi\|\|^2 &\leq \tilde{\mathcal{B}}_\varepsilon(\xi,\xi) = -\tilde{\mathcal{B}}_\varepsilon(\eta,\xi) \\ &= -\int_{\mathcal{T}_{cG}} \varepsilon \nabla_h \eta \cdot \nabla_h \xi + (b \cdot \nabla_h \eta)\xi + c\eta\xi \, d\boldsymbol{x} + \int_J b \cdot [\![\eta]\!]\xi^{\text{in}} \, ds. \end{aligned}$$

Due to Assumption 2.33 we have $\xi^{\text{in}} = \xi^{\mathcal{D}}$, the trace from the dG side of J, and $[\![\eta]\!] = \eta^{\mathcal{C}} n^{\mathcal{C}}$, the trace and normal from the cG side of J. We split each of the terms using Young's inequality, giving

$$\begin{aligned} \tfrac{1}{4}\|\|\xi\|\|^2 &\leq 2\varepsilon \|\nabla_h \eta\|^2_{L^2(\mathcal{T}_{cG})} + \frac{\varepsilon}{8}\|\nabla_h \xi\|^2_{L^2(\mathcal{T}_{cG})} + \frac{4}{\rho}\|b \cdot \nabla_h \eta\|^2_{L^2(\mathcal{T}_{cG})} + \frac{\rho}{16}\|\xi\|^2_{L^2(\mathcal{T}_{cG})} \\ &\quad (4.5) \qquad + \frac{4}{\rho}\|c\|^2_{L^\infty(\Omega)}\|\eta\|^2_{L^2(\mathcal{T}_{cG})} + \frac{\rho}{16}\|\xi\|^2_{L^2(\mathcal{T}_{cG})} + \int_J (b \cdot n^{\mathcal{C}} \eta^{\mathcal{C}})\xi^{\mathcal{D}} \, ds. \end{aligned}$$

For the final term we note that ξ is a polynomial and so using Young's inequality and a trace and inverse inequality (with constant C_{ti}) gives

$$(4.6) \qquad \int_J (b \cdot n^{\mathcal{C}} \eta^{\mathcal{C}})\xi^{\mathcal{D}} \, ds \leq \frac{4C_{\text{ti}}\|b\|^2_{L^\infty(\Omega)}}{h_e \rho}\|\eta^{\mathcal{C}}\|^2_{L^2(J)} + \frac{\rho}{16}\|\xi\|^2_{L^2(\mathcal{T}_{dG})}.$$

We combine (4.5) and (4.6) to hide all terms of ξ under the norm on the left-hand side of (4.5). Using (3.3) for the terms of η and a trace inequality gives

$$\rho\|\xi\|^2_{L^2(\mathcal{T}_{cG})} \leq \|\|\xi\|\|^2 \lesssim (\varepsilon h^2_{\mathcal{T}_{cG}} + h^4_{\mathcal{T}_{cG}} + h^2_{\mathcal{T}_{cG}})\|u_0\|^2_{H^2(\mathcal{T}_{cG})} \lesssim h^2_{\mathcal{T}_{cG}}\|u_0\|^2_{H^2(\mathcal{T}_{cG})}$$

and, by an inverse inequality, $\|\xi\|^2_{H^1(\mathcal{T}_{cG})} \lesssim 1$. Now the result follows from the stability of the Scott-Zhang operator. $\qquad \square$

5 Inf-Sup Condition

The following theorem is an adaptation of related stability bounds in [5] and [2] to fit the above assumptions. Although the verification of the below inf-sup condition follows the overall structure in [5], we state it here in detail as the present analysis extends the scope to non-constant advection coefficients via the incorporation of $\Pi_{\mathcal{D}}$ as [5]. Moreover, it deals with the modification of the bilinear form on J and it only has streamline control on the $\mathcal{T}_{dG}$ side. It is helpful to recall that $\Pi_{\mathcal{D}}v|_{\mathcal{T}_{cG}} = 0$ for any v.

Theorem 5.1. *There exists a positive constant Λ_{is} which is independent of h and ε but may depend on the polynomial degree, σ, and the constants in (2.25) and (2.26) such that:*

$$(5.2) \qquad \inf_{v \in V_{cdG}} \sup_{\hat{v} \in V_{cdG}} \frac{\tilde{\mathcal{B}}_\varepsilon(v,\hat{v})}{\|v\|_S \|\hat{v}\|_S} \geq \Lambda_{\text{is}}.$$

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Proof. For an arbitrary $v \in V_{\text{cdG}}$, we define

$$(5.3) \qquad \hat{v} := v + \gamma\, v_S, \qquad v_S := \sum_{E \in \mathcal{T}_h} \tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v),$$

where γ is a positive parameter at our disposal and τ_E is defined in (2.28). Note that through the definition of $\Pi_{\mathcal{D}}$ we have $\hat{v}, v_S \in V_{\text{cdG}}$. Theorem 5.1 is equivalent to showing the following two results:

$$(5.4) \qquad \|\hat{v}\|_S \lesssim \|v\|_S,$$

$$(5.5) \qquad \tilde{\mathcal{B}}_{\varepsilon}(v, \hat{v}) \gtrsim \|v\|_S^2.$$

Consider first (5.4). We examine each term of $\|v_S\|_S^2$ in turn. We have

$$
\begin{aligned}
(5.6) \qquad \sum_{E \in \mathcal{T}_h} \varepsilon |v_S|_{H^1(E)}^2 &\lesssim \sum_{E \in \mathcal{T}_h} \varepsilon h_E^{-2} \|\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2 \\
&\leq \sum_{E \in \mathcal{T}_h} \tau \tau_E \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2 \lesssim \|v\|_S^2.
\end{aligned}
$$

Also

$$(5.7) \qquad \|r^{1/2} v_S\|_{L^2(\Omega)}^2 \leq \|r\|_{L^\infty(\Omega)} \sum_{E \in \mathcal{T}_h} \tau_E^2 \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2 \lesssim \|v\|_S^2.$$

For the terms on the edges we use (2.26). This gives

$$(5.8) \qquad \sum_{e \in \mathcal{E}_h} \||b \cdot n|^{1/2} [\![v_S]\!]\|_{L^2(e)}^2 \lesssim \sum_{E \in \mathcal{T}_h} \|b\|_{L^\infty(\Omega)} \tau_E^2 h_E^{-1} \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2 \lesssim \|v\|_S^2.$$

Similarly,

$$(5.9) \qquad \sum_{e \in \mathcal{E}_h} \frac{\sigma \varepsilon}{h_e} \|[\![v_S]\!]\|_{L^2(e)}^2 \lesssim \sum_{E \in \mathcal{T}_h} \tau_E^2 \frac{\sigma \varepsilon}{h_E^2} \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2 \lesssim \|v\|_S^2.$$

The final term of the streamline norm gives

$$
\begin{aligned}
\sum_{E \in \mathcal{T}_h} \tau_E \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v_S)\|_{L^2(E)}^2 &\leq \sum_{E \in \mathcal{T}_h} \tau_E \|b \cdot \nabla_h (\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v))\|_{L^2(E)}^2 \\
&\lesssim \sum_{E \in \mathcal{T}_h} \tau_E^3 \|b\|_{L^\infty(E)}^2 h_E^{-2} \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2 \lesssim \|v\|_S^2.
\end{aligned}
$$

Combining the above results we have $\|v_S\|_S^2 \lesssim \|v\|_S^2$. Using a triangle inequality we find

$$\|\hat{v}\|_S \leq \|v\|_S + \gamma \|v_S\|_S \leq C(\tau, \sigma, \gamma)\|v\|_S,$$

which concludes the proof of (5.4).

To prove (5.5) first consider the advection and reaction terms of the norm. Using the linearity of $\mathcal{B}_{ar}$ we have $\mathcal{B}_{ar}(v, \hat{v}) = \mathcal{B}_{ar}(v, v) + \gamma \mathcal{B}_{ar}(v, v_S)$. The second term is given by

$$
\begin{aligned}
\mathcal{B}_{ar}(v, v_S) = &\sum_{E \in \mathcal{T}_h} \int_E c v (\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)) + (b \cdot \nabla_h v)(\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)) \, d\boldsymbol{x} \\
&- \sum_{e \in \mathcal{E}_h^o} \int_e b \cdot [\![v]\!] (\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v))^{\text{in}} \, ds - \sum_{e \in \Gamma^{\text{in}}} \int_e (b \cdot n) v (\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)) \, ds.
\end{aligned}
$$

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Using the properties of $\Pi_{\mathcal{D}}$ given in (2.23) the second term above becomes

$$
\text{(5.10)} \quad \sum_{E \in \mathcal{T}_h} \int_E (b \cdot \nabla_h v)(\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)) \, \mathrm{d}\boldsymbol{x} = \sum_{E \in \mathcal{T}_h} \int_E \tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v) \Pi_{\mathcal{D}}(b \cdot \nabla_h v) \, \mathrm{d}\boldsymbol{x}
$$
$$
= \sum_{E \in \mathcal{T}_h} \tau_E \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2.
$$

Using Young's inequality we have

$$
\left| \sum_{E \in \mathcal{T}_h} \int_E cv(\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)) \, \mathrm{d}\boldsymbol{x} \right| \leq \|c\|_{L^\infty(\Omega)} \sum_{E \in \mathcal{T}_h} \frac{1}{2} \|v\|_{L^2(E)}^2 + \frac{1}{2} \tau_E^2 \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2
$$

and, where C arises from a trace inequality and the number of edges per element,

$$
- \sum_{e \in \mathcal{E}_h^o} \int_e b \cdot [\![v]\!](\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v))^{\mathrm{in}} \, \mathrm{d}s - \sum_{e \in \Gamma^{\mathrm{in}}} \int_e (b \cdot n) v(\tau_E \Pi_{\mathcal{D}}(b \cdot \nabla_h v)) \, \mathrm{d}s
$$
$$
\leq \sum_{e \in \mathcal{E}_h} \frac{C\lambda}{2} \||b \cdot n|^{1/2} [\![v]\!]\|_{L^2(e)}^2 + \sum_{E \in \mathcal{T}_h} \frac{\tau_E \tau}{2\lambda} \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(E)}^2.
$$

In conclusion, using (2.37), we have

$$
\text{(5.11)} \quad \mathcal{B}_{ar}(v, \hat{v}) \geq \left(\rho - \frac{\gamma \|c\|_{L^\infty(\Omega)}}{2} \right) \sum_{E \in \mathcal{T}_h} \|v\|_{L^2(E)}^2 + \left(\frac{1}{2} - \frac{\gamma C\lambda}{2} \right) \sum_{e \in \mathcal{E}_h} \||b \cdot n|^{1/2} [\![v]\!]\|_{L^2(e)}^2
$$
$$
+ \gamma \sum_{E \in \mathcal{T}_h} \left(\tau_E - \frac{\tau_E^2}{2} - \frac{\tau_E \tau}{2\lambda} \right) \|\Pi_{\mathcal{D}}(b \cdot \nabla_h v)\|_{L^2(\Omega)}^2.
$$

Recall that $\|h_E/b\|_{L^\infty(\mathcal{T}_{\mathrm{dG}})} \leq 1$ by Assumption 2.12, which implies $\tau_E \leq 1$ for all $E \in \mathcal{T}_h$. For general v, all terms on the right-hand side of (5.11) are positive, provided λ is large and γ is small enough.

Equation (2.22) ensures the continuity of $\tilde{\mathcal{B}}_d$ with respect to $\|\cdot\|_d$; thus

$$
\text{(5.12)} \quad \tilde{\mathcal{B}}_d(v, \hat{v}) \leq C_1 \|v\|_d \|\hat{v}\|_d,
$$

for some $C_1 > 0$. Recalling (5.6) and (5.9), it is clear that $\|v_S\|_d \leq C_2 \|v\|_d$, for some $C_2 > 0$. Hence

$$
\text{(5.13)} \quad \tilde{\mathcal{B}}_d(v, \hat{v}) = \tilde{\mathcal{B}}_d(v, v) + \gamma \tilde{\mathcal{B}}_d(v, v_S) \geq \tfrac{1}{4} \|v\|_d^2 - \gamma C_1 \|v\|_d \|v_S\|_d.
$$

Thus, if $\gamma < C_1 C_2 / 8$, then $\tilde{\mathcal{B}}_d(v, \hat{v}) \geq \tfrac{1}{8} \|v\|_d^2$ which, combined with (5.11), gives (5.5). $\square$

6 Stability of the Decoupled and Standard Approximations

We saw that, under a set of suitable assumptions, the decoupled approximation satisfies the stability bounds:

$$
\text{(6.1)} \quad \|\tilde{v}_h\|_{H^1(\mathcal{T}_{\mathrm{cG}})} \lesssim 1, \qquad \|\tilde{v}_h\|_S \lesssim \|f\|_{L^2(\Omega)}.
$$

The first bound is a consequence of Theorem 3.4 and Lemma 4.4, the second of Theorem 5.1. So while one has streamline-diffusion stability on $\mathcal{T}_{\mathrm{dG}}$, an even stronger bound is available on $\mathcal{T}_{\mathrm{cG}}$ under the aforementioned assumptions. We now derive a stability result for the cdG method. We require that the geometry of the interface J does not become significantly more complicated as the mesh is refined. More precisely, we require the boundedness of the trace operator.

Theorem 6.2. *Suppose that the operator norm of the trace $H^1(\mathcal{T}_{cG}) \to L^2(J)$ is bounded independently of h. Then, the cdG approximation v_h is stable in the sense that*

$$h_{\mathcal{T}_{cG}}\|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{cG})} + \|v_h\|^2_S \lesssim 1 + \|f\|^2_{L^2(\Omega)}.$$

Proof. Set $\zeta := v_h - \tilde{v}_h$. Using the coercivity of $\mathcal{B}_\varepsilon$, Galerkin orthogonality and the norm of the trace $H^1(\mathcal{T}_{cG}) \to L^2(J)$, we have

$$\frac{1}{4}|||\zeta|||^2 \le \mathcal{B}_\varepsilon(\zeta, \zeta) = \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_h, \zeta) - \mathcal{B}_\varepsilon(\tilde{v}_h, \zeta) + \mathcal{B}_\varepsilon(v_h, \zeta) - \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_h, \zeta)$$

$$= \tilde{\mathcal{B}}_\varepsilon(\tilde{v}_h, \zeta) - \mathcal{B}_\varepsilon(\tilde{v}_h, \zeta)$$

$$= \varepsilon \int_J \{\nabla_h \tilde{v}_h\} \cdot [\![\zeta]\!] + \{\nabla_h \zeta\} \cdot [\![\tilde{v}_h]\!] - \frac{\sigma}{h_e}[\![\tilde{v}_h]\!] \cdot [\![\zeta]\!] \, ds$$

$$\lesssim \varepsilon \Big(h_{\mathcal{T}_{cG}} \|\nabla_h \tilde{v}_h\|^2_{L^2(\Omega)} \cdot \frac{\varepsilon\sigma}{h_e^{3/2}} \|[\![\zeta]\!]\|^2_{L^2(J)} + \|\nabla_h \zeta\|^2_{L^2(\Omega)} \cdot \frac{\varepsilon\sigma}{h_e^{1/2}} \|[\![\tilde{v}_h]\!]\|^2_{L^2(J)}$$

$$+ \int_J \frac{\sigma}{h_e} [\![\tilde{v}_h]\!] \cdot [\![\zeta]\!] \, ds \Big)$$

$$\lesssim \Big(\varepsilon h_{\mathcal{T}_{cG}} \|\nabla_h \tilde{v}_h\|^2_{L^2(\Omega)} + \frac{\varepsilon\sigma}{h_e^{1/2}} \|[\![\tilde{v}_h]\!]\|^2_{L^2(J)} \Big)^{1/2} \Big(\varepsilon\|\nabla_h \zeta\|^2_{L^2(\Omega)} + \frac{\varepsilon\sigma}{h_e^{3/2}} \|[\![\zeta]\!]\|^2_{L^2(J)} \Big)^{1/2}$$

and thus, using Assumption 2.33 for $\varepsilon\sigma h_e^{-3/2}$,

$$(6.3) \qquad\qquad |||\zeta|||^2 \lesssim \varepsilon h_{\mathcal{T}_{cG}} \|\nabla_h \tilde{v}_h\|^2_{L^2(\Omega)} + \frac{\varepsilon\sigma}{h_{\mathcal{T}_{cG}}^{1/2}} \|[\![\tilde{v}_h]\!]\|^2_{L^2(J)}.$$

Dividing through by $h_{\mathcal{T}_{cG}}$ and using an inverse inequality on $\rho\|\zeta\|_{L^2(E)}$ gives

$$(6.4) \qquad h_{\mathcal{T}_{cG}} \|\nabla_h \zeta\|^2_{L^2(\mathcal{T}_{cG})} \lesssim h_{\mathcal{T}_{cG}}^{-1} |||\zeta|||^2 \lesssim \varepsilon\|\nabla_h \tilde{v}_h\|^2_{L^2(\Omega)} + \frac{\varepsilon\sigma}{h_{\mathcal{T}_{cG}}^{3/2}} \|[\![\tilde{v}_h]\!]\|^2_{L^2(J)}.$$

Using Assumptions 2.12 and 2.33, as well as (6.1), we bound each of the terms in (6.4). Using a triangle inequality on $\|\nabla\zeta\|_{L^2(\mathcal{T}_{cG})}$ we conclude that

$$h_{\mathcal{T}_{cG}}\|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{cG})} \lesssim 1.$$

To show that $\|v_h\|_S$ is bounded we establish an inf-sup condition for $\mathcal{B}_\varepsilon$. Indeed, (5.4) may be used without change. It remains to transfer (5.5) to $\mathcal{B}_\varepsilon$. The inequality (5.11) is still available as the discretesation of the lower-order terms did not change. Different is that we now use $\mathcal{B}_d(v, \hat{v}) \le C_1|||v||| \cdot |||\hat{v}|||$ in place of (5.12), justified by Assumption 2.33. Appealing to (5.6)–(5.9), one has $|||v_S||| \le C_2|||v|||$ for some $C_2 > 0$. Hence

$$(6.5) \qquad\qquad \tilde{\mathcal{B}}_d(v, \hat{v}) = \tilde{\mathcal{B}}_d(v, v) + \gamma \tilde{\mathcal{B}}_d(v, v_S) \ge \frac{1}{4}\|v\|^2_d - \gamma C_1|||v||||||v_S|||.$$

For $\gamma C_1 C_2$ small enough and λ sufficiently large, $\gamma C_1|||v||||||v_S|||$ is bounded by $\frac{1}{8}\|v\|^2_d + \frac{1}{2}\mathcal{B}_{ar}(v, \hat{v})$, using again the positivity of the terms in (5.11). □

Remark 6.6. *Due to Assumptions 2.12 and 2.33 the above stability bound is valid for the regime $\varepsilon \lesssim h_E^{3/2}\|b\|_{L^\infty(\Omega)}$. For completeness we briefly outline here how $\sqrt{h}\|\cdot\|_{H^1(\Omega)}$ stability of the cdG method is established if $\varepsilon \gtrsim h_E^{3/2}\|b\|_{L^\infty(\Omega)}$. The stability proof is in this case easier because the mesh Péclet number is smaller. If Ω is smooth or convex and the coefficients have sufficient regularity then u is in $H^2(\Omega)$. Indeed for $\varepsilon \ge \frac{1}{2}h_E^{3/2}\|b\|_{L^\infty(\Omega)}$ the H^2 norm of u is uniformly bounded in ε. Suppose that the mesh is quasi-uniform. By Ceà's Lemma, with $h := \max_E h_E$, a standard argument gives*

$$\|u - v_h\|_{H^1(\Omega)} \lesssim (1 + \varepsilon^{-1})h\|u\|_{H^2(\Omega)} \lesssim (h + h^{-1/2})\|u\|_{H^2(\Omega)},$$
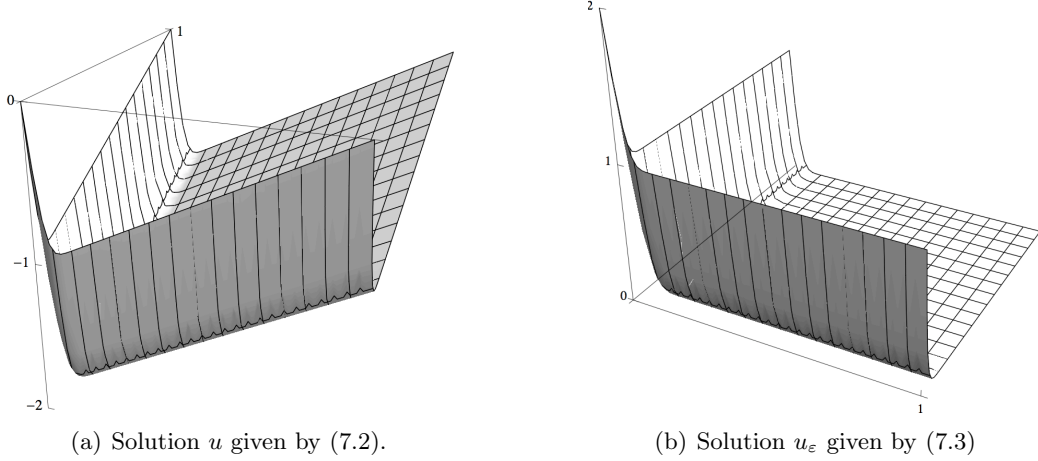
and thus $\sqrt{h}\|u - v_h\|_{H^1(\Omega)} \lesssim 1$.

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(a) Solution u given by (7.2).   (b) Solution u_ε given by (7.3)

Figure 7.1: Solution u and u_ε for $\varepsilon = 10^{-3}$.

7 Numerical Experiment

Let $\Omega = (0,1)^2$. We seek to solve

$$(7.1) \qquad\qquad -\varepsilon \Delta u + (-x, -y) \cdot \nabla u = -x - y$$

with Dirichlet boundary conditions chosen such that the solution is given by

$$(7.2) \qquad\qquad u(x,y) = x + y - \frac{\mathrm{Erf}\left(x/\sqrt{2\varepsilon}\right) + \mathrm{Erf}\left(y/\sqrt{2\varepsilon}\right)}{\mathrm{Erf}\left(1/\sqrt{2\varepsilon}\right)}$$

where Erf is the error function defined by $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, \mathrm{d}t$. For $0 < \varepsilon \ll 1$ this problem exhibits an exponential boundary layer along the outflow boundaries $x = 0$ and $y = 0$ of width $\mathcal{O}(\sqrt{\varepsilon})$.

Away from the layers the boundary conditions on the inflow boundaries $x = 1$ and $y = 1$ are well approximated by $y - 1$ and $x - 1$ respectively. The hyperbolic solution with these boundary conditions is given by $u_0(x,y) = x + y - 2$. This gives

$$(7.3) \qquad\qquad u_\varepsilon(x,y) = 2 - \frac{\mathrm{Erf}\left(x/\sqrt{2\varepsilon}\right) + \mathrm{Erf}\left(y/\sqrt{2\varepsilon}\right)}{\mathrm{Erf}\left(1/\sqrt{2\varepsilon}\right)}.$$

We plot (7.2) and (7.3) for $\varepsilon = 10^{-3}$ in Figure 7.1; note that away from the layers the solution u_ε is close to zero.

We let Ω_{cG} be a set of the type $(1 - \delta, 1)^2$, $0 < \delta < 1$. For each δ the supremum

$$\sup_{\varepsilon \in (0, \varepsilon_{\max}]} \|u_\varepsilon\|_{H^2(\Omega_{\mathrm{cG}})}$$

is finite. The dependence of $\|u_\varepsilon\|_{H^2(\Omega_{\mathrm{cG}})}$ with respect to δ and to ε is illustrated in Figure 7.2.

For this example $c - \frac{1}{2}\nabla \cdot b = 1$, so Assumption 2.9 is satisfied. Further, we fix $\varepsilon = 10^{-6}$ and consider uniform square meshes of edge length 2^{-5} so that Assumption 2.12 is also satisfied as the smallest local mesh Péclet number being 488.28.

We define $\mathcal{T}_{\mathrm{cG}} = [1 - \delta_h, 1]^2$, where $\delta_h = m2^{-5}$, $m \in \{0, \ldots, 32\}$. Note that, having fixed the mesh, δ_h is a discrete parameter. The interface J is composed of the edges lying
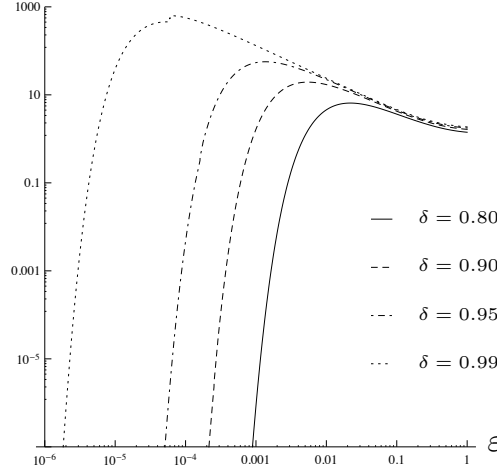
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Figure 7.2: $\|u_\varepsilon\|_{H^2(\Omega_{\mathrm{cG}})}$ for different values of δ.
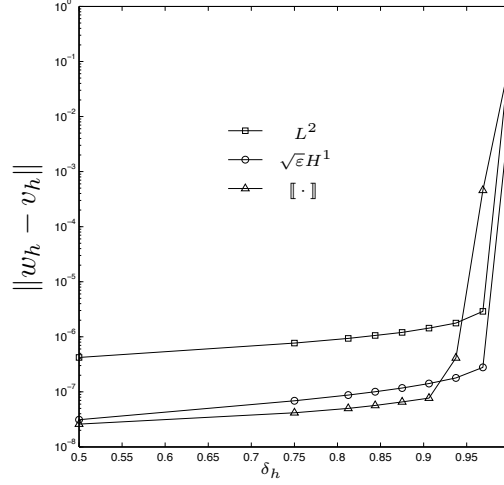

Figure 7.3: Difference between cdG and dG solutions.

on the lines $y = \delta_h$ for $x \geq \delta_h$ and $x = \delta_h$ for $y \geq \delta_h$. The smallest value of $b \cdot n$ is δ_h occuring on the edges containing the point (δ_h, δ_h). Thus, in this case, Assumption 2.33 reads $\frac{1}{4}mh > \varepsilon\frac{\sigma}{h^{3/2}}$ and is satisfied for all $m \in \{1, \ldots, 32\}$ (note that this assumption is trivially satisfied when $m = 0$) for this choice of ε and h if $\sigma < 200$; in the shown computations $\sigma = 10$.

In Figure 7.3 we plot the $L^2(\Omega)$ norm, $\sqrt{\varepsilon}$ weighted $H^1(\mathcal{T}_h)$ semi-norm, and L^2 norm of the jumps on $\mathcal{E}_h$(represented by $[\![\cdot]\!]$) for both the difference in the dG and cdG approximations and the error in the cdG approximation. Note from Figure 7.3 that the difference in the approximations increases only very slowly until the final data points (where $\mathcal{T}_{\mathrm{cG}} \approx \mathcal{T}_h$). When the continuous region covers the layer, non-physical oscillations pollute the approximation in function of δ_h.

In Table 7.1 we show the number of degrees of freedom (dofs) as the continuous region is increased. Reducing the degrees of freedom to approximately 30% of the dG method degrees of freedom results in only a very slight difference in the norm, thus showing that a considerable saving can be made without compromising stability.

We finally remark that, at least for the example considered here, the choice of $\mathcal{T}_{\mathrm{cG}}$

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$1 - \delta$	dofs	% of dG dofs	$\sqrt{\varepsilon}\|\nabla_h(w_h - v_h)\|_{H^1(\Omega)}$
dG	4096	100	0.0
8×2^{-5}	3361	82.1	3.1157e-08
16×2^{-5}	2417	59.0	6.8911e-08
24×2^{-5}	2121	51.8	8.7544e-08
30×2^{-5}	1457	35.6	1.7934e-07
31×2^{-5}	1276	31.2	2.7896e-07
cG	1089	26.6	1.2444e-02

Table 7.1: Degrees of freedom with $\varepsilon = 10^{-6}$.

leaving one layer of elements at the outflow boundary is optimal. Indeed, adding even a single element to the $\mathcal{T}_{\mathrm{cG}}$ region results in oscillations polluting the solution. For example, for

$$\mathcal{T}_{\mathrm{cG}} = [2^{-5}, 1]^2 \cup ([0.5, 0.5 + 2^{-5}] \times [0, 2^{-5}]),$$

i.e., adding a sinlge element to $\mathcal{T}_{\mathrm{cG}}$ halfway along the x-axis, results in

$$\|w_h - v_h\|_{L^2(\Omega)} = 4.7966 \times 10^{-2},$$
$$\sqrt{\varepsilon}\|\nabla_h(w_h - v_h)\|_{L^2(\Omega)} = 4.5008 \times 10^{-3},$$

a significant increase on the norms for $\mathcal{T}_{\mathrm{cG}} = [2^{-5}, 1]^2$. Notice that this choice of $\mathcal{T}_{\mathrm{cG}}$ violates Assumption 2.33.

8   Acknowledgements:

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